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CARNAP'S PROBLEM: WHAT IS IT LIKE TO BE A NORMAL INTERPRETATION OF CLASSICAL LOGIC?*

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Abstract
Carnap in the 1930s discovered that there were non-normal interpretations of classical logic - ones for which negation and conjunction are not truth-functional so that a statement and its negation could have the same truth value, and a disjunction of two false sentences could be true. Church argued that this did not call for a revision of classical logic. More recent writers seem to disagree. We provide a definition of "non-normal interpretation" and argue that Church was right, and in fact, the existence of non-normal interpretations tells us something important about the conditions of extensionality of the classical logical operators.

1. Carnap's Problem

In the decade from the early thirties to the mid fifties, there was a brief and scattered discussion of a problem raised by B.A. Bernstein (1932), R.Carnap (1943), and A. Church (1944, 1956) of what has now been referred to referred to as "Carnap's Problem". Carnap discovered the existence of what Church later called "non-normal interpretations" of sentential classical logic, and first –order logic. Church's major criticism of Carnap's reformulation of sentential logic was that it essentially incorporated semantical assumptions into what was supposed to be a syntactically presented formulation of the logic.

In what follows, I shall consider only sentential logics. Roughly speaking Carnap took interpretations to be truth-value assignments (He called them "interpretations.") which assigned truth to all theorems, and which respected deducibility—that is, if some collection of sentences is true under an interpretation τ, then any sentence deducible from those sentences is also true under τ. What Carnap discovered was that there were interpretations of the classical sentential calculus which assigned the same truth value to statements as well as

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their negations, and interpretations which assigned “true” to a disjunction while also assigning “false” to all of its disjuncts.

Church thought was that there was no need to correct the formulation of the logic. There was no “deficiency” in the formalization, and no need to seek a “fuller formalization”. Presumably if one just avoided the use of these non-normal interpretations, there would not be any mismatch between the proof-theoretic (syntactic) deductive presentation of the system, and the usual (semantic) truth-tables for the logical connectives.

In the more recent literature devoted to Carnap’s discovery, the issue takes on a more serious cast. If one thought that the truth-tables provided the meaning of the logical connectives (I do not), then if the proof-theoretic formulation does not match up with the tables then one might put the significance of Carnap’s discovery as showing that the (deductive) rules of inference do not determine the meanings of logical constants (Raatikainen (2008)). Other writers (Murzi & Hjortland (2010)) have taken the moral to be one that concerns inferentialism, and a problem concerning a special kind of categoricity. Shoesmith and Smiley (1978) have explored specific examples of non-normal interpretations from the vantage of multiple -conclusion logics, using essentially a four-element Boolean algebra, and Smiley (1996), Incurvati & Smith (2009) have considered embedding the sentential calculus in a system with rules of rejection and acceptance (a “fuller fomalization”?) to eliminate the mismatch. Rumfitt (1997 & 2000) and a host of other logicians have made their own case for understanding the import of these strange truth-value assignments.

In addition to all of these, I now wish to reconsider “Carnap’s challenge” and its import from a more general “structuralist” vantage. This kind of approach has been explained at length in Koslow (1992), and somewhat differently but in lesser length in Koslow (1999). The following discussion however is intended to be self contained.

2. The more general structuralist background

To indicate the generality of "Carnap's Problem" we shall use the notion of an *implication structure* $\mathfrak{I} = \langle S, \Rightarrow \rangle$, where S is any non-empty set, and “$\Rightarrow$” is an *implication relation.* That is, any relation on S satisfying the following six slightly redundant conditions:

(1) ***Refexivity***: $A \Rightarrow A$, for all A in S.

(2) ***Projection***: $A_1,A_2,...,A_n \Rightarrow A_k$, for any k= 1,...,n.

(3) ***Simplification*** (sometimes called Contraction): If $A_1,A_1,A_2,...,A_n \Rightarrow B$, then $A_1,A_2,...,A_n \Rightarrow B$, for all A_i and B in S.

(4) ***Permutation***: If $A_1,A_2,...,A_n \Rightarrow B$, then $A_{f(1)},A_{f(2)},...,A_{f(n)} \Rightarrow B$, for any permutation f of {1,2,...,n}.

(5) *Dilution*: If $A_1,A_2,...,A_n \Rightarrow B$, then $A_1,A_2,...,A_n,C \Rightarrow B$, for all A_i,B, and C in S.

(6) ***Cut***: If$A_1,A_2,...,A_n \Rightarrow B$, and B, $B_1,B_2,...,B_m \Rightarrow C$, then $A_1,A_2,...,A_n,B_1,B_2,...,B_m \Rightarrow C$.

These conditions are of course those which G.Gentzen put forward as the structural conditions for implication. We understand them in a very general sense as giving a story about implication that does not appeal to truth or any other familiar semantical concept. And it is conspicuous that the story is told without any appeal to the logical operators. They (that is, the operators of conjunction, disjunction, the conditional, negation, and universal and existential quantification), as it turns out can all be defined in terms of implication. It is also a feature of the structuralist story that the set S is not restricted to syntactically presented elements. In that lies the generality of this way of looking at things. Nevertheless it is a generality that will not be used in the following.

Here are a few definitions that we shall need in order to show how, despite the absence of apparently semantical concepts in this story, we can in fact define the concept of truth-value assignments (valuations), and obtain with them, a remarkable completeness theorem for the theory of implications given by (1) – (6). This will allow us to introduce the semantic notion of a valuation in the structuralist setting.

(i) A ***bisection*** on S is any ordered pair T = <K, L> where K and L are non-empty subsets of S that are disjoint , and whose union is S.

(ii) Let T = <K, L> be a bisection on S. Then $\Rightarrow^T$ is the corresponding bisection implication relation defined as follows:

$A_1, A_2, ..., A_n \Rightarrow^T$ B if and only if some A_i is in K or B is in L.

(iii) Let $\Im$ = <S, ⇒ > be an implication structure, then any subset R of S is ***strongly closed*** under the implication relation if and only if whenever several members of R together imply A then A also belongs to R. We shall say that any bisection T = <K, L> on an implication structure is a strong bisection on $\Im$ if and only if L is strongly closed under the implication relation ⇒ of the structure.

3. Truth-value assignments (valuations)

We can now show several interesting facts about strong bisections, on the basis of which we can define the notion of a ***truth-value interpretation*** on arbitrary implication structures.

(iv) Let $\Im$ = <S, ⇒ > be an implication structure. Then for any strong bisection implication relation $\Rightarrow^T$ on $\Im$, we have $\Rightarrow \subseteq \Rightarrow^T$. That is, for any structure, every strong bisection implication $\Rightarrow^T$ on it extends ⇒.

(v) The strong bisection implications are maximal. That is, if $\Rightarrow^T$ and $\Rightarrow^{T*}$ are strong bisection implication relations on $\Im$, and $\Rightarrow^T \subseteq \Rightarrow^{T*}$, then $\Rightarrow^T = \Rightarrow^{T*}$.

We now come to the basic result that allows the introduction of truth-value interpretations on implication structures:

(vi) **Lindenbaum-Scott Completeness I (**Scott, 1974). Let $\Im$ be any non-trivial implication structure (there are at least two elements of it neither of which implies the other). Then $A_1, A_2, ..., A_n \Rightarrow$ B if and only if $A_1, A_2, ..., A_n \Rightarrow^T$ B, for all strong bisection relation $\Rightarrow^T$ on the structure.

Another way of stating this result is that if we define an implication structure as complete if and only every member of it is either a thesis (it is implied by everything in the structure, or it is an antithesis, it implies everything in the structure, then the Lindenbaum-Scott Theorem says that every non-trivial implication relation is the intersection of all complete implication relations that extend it.

> Simple proof: For any (non-trivial) implication structure $\mathfrak{I}$ = <S, ⇒ >, clearly every bisection implication relation $\Rightarrow^T$ extends the implication relation ⇒, because L is strongly closed under the implication relation ⇒. The converse is fairly straightforward: Suppose that $A_1, A_2, ..., A_n \Rightarrow^T B$ for all bisection implication relations $\Rightarrow^T$, but $A_1, A_2, ..., A_n \Rightarrow B$ fails. We define a strong bisection T* = <K, L> as follows: Let L be the set of all members C of S such that $A_1, A_2, ..., A_n \Rightarrow C$. L is strongly closed and is also non-empty since it contains all the A_i. Let K be the rest of S, that is, all C such that $A_1, A_2, ..., A_n \Rightarrow C$ fails. It is non-empty since B is in it. So T is a strong bisection on the structure. Therefore, by hypothesis, $A_1, A_2, ..., A_n \Rightarrow^{T*} B$. Therefore some A_i is in K or B is in L. But none of the A_i are in K so B is in L. But that is impossible. Consequently, $A_1, A_2, ..., A_n \Rightarrow B$.

We can now define the notion of a truth-value assignment for arbitrary (non-trivial) implication structures. Simply stated, truth-value assignments (valuations) on implication structures are uniquely associated with strong bisections on those structures. That is

If $\mathfrak{I}$ = <S, ⇒ > is an implication structure, then any truth-value assignment on it is a function τ associated with a strong bisection T = <K, L> on it such that for any A in S,

τ(A) = t, if A is in L, and

τ(A) = f, if A is in K.

With this notion of a valuation in place, the Lindenbaum-Scott theorem can be stated in a way that is truly a completeness result:

> **Lindenbaum-Scott Completeness II.** Let $\mathfrak{I}$ be any non-trivial implication structure. Then $A_1, A_2, ..., A_n \Rightarrow B$ if and only if for all strong bisection relations τ on the structure, if τ(A_i) = t for all A_i , then τ(B_i) = t. That is the implication relation on the structure holds if and only if every valuation that makes each of the premises true, also makes the conclusion true.

A few observations are in order. In some of the recent literature on Carnap's Problem, much has rested on the truism that if the premises imply a conclusion, then if the premises are true under a valuation, then the conclusion has got to be true as well. Some distinguished logicians like J. Myhill, and J. Corcoran have had their doubts whether this is even correct, but the simple proof shows that something like it is correct. However, a glance at the simple proof of completeness shows it to be a result which holds without assuming how the valuations behave with respect to the logical operators. Thus on the present story, it is not at all plausible, that without some additional assumptions such matters as how valuations distribute over conjunctions, disjunctions, negations and conditionals will be settled, or indeed whether they can be settled in any but special cases.

Nevertheless, it is this theorem that motivates our taking these valuations as truth-value assignments. It is what one sees in the usual classical case, only instead of the usual truth values, we take truth (falsity) of an interpretation to be just membership in the sets L and K of the associated strong bisection (this is an insight which is due to D. Scott (1974).

It is worth recalling that this notion of a truth-value assignment relies only on the notion of an implication relation as we described it using Gentzen Structural conditions. Those conditions for implication made no appeal to any notion of truth, or truth-value assignments. So the definition of these valuations does not rely on some hidden semantic devices.

Nevertheless, there is a fair amount of semantic information that can be gleaned. It can be shown that

For any conjunction [$A \wedge B$ is in L if and only if A is in L and B is in L].

For negation, [If A in in L then $\neg A$ is in K] (but not conversely.

For disjunctions, [If A is in L or B is in L, then $(A \vee B)$ is in L] (but not conversely), and

For conditionals [If $(A \rightarrow B)$ is in L, then either A is in K or B is in L] (but not conversely).

So if we want more, we need to require more than just truth-preservation under all valuations.

4. Carnap's Problem, non-normal truth-value interpretations, and the logical operators

With the Lindenbaum-Scott theorem in place we are in a position to reconsider Carnap's Problem. It involves the observation that there are certain truth-value valuations which satisfy the condition that any sentence that is implied by sentences that are all assigned the value "true", will also be assigned the value "true", which, nevertheless, lead to unwanted consequences for the logical operators. It will turn out that there are assignments that will assign the value "f" both to a sentence and it's negation, and some that will also assign "t" to a disjunction of sentences each of which has been assigned "f". There's something peculiar about the examples of non-normal valuations that Carnap, and Church provided. But we shall see with the help of other examples that there are such non-normal valuations on implication structures even when the notion of a truth-value assignment is given the clear foundation provided by the Lindenbaum-Scott theorem. The "Carnap" phenomenon is real.

It is instructive to note that there are cases of implication structures with implication relations that do not give rise to a Carnap Problem; all truth-value assignments for the logical operators behave in the expected way. Let S be the sentences of the classical sentential calculus, and let the implication relation be given by the bisection implication relation $\Rightarrow^{T}$, where T is a strong bisection <K, L> (that is, L is closed under the implication relation $\Rightarrow^{T}$). It is easy to see that the following holds:

(1) $(A \wedge B)$ is in L (t) if and only if A is in L (t) and B is in L (t).

(2) $\neg A$ is in L (t) if and only if A is in K (f).

(3) $(A \vee B)$ is in L (t) if and only if A is in L (t) or B is in L (t).

(4) $(A \rightarrow B)$ is in L (t) if and only if A is in K (f) or B is in L (t).

That is, all the operators behave in the familiar extensional pattern for this particular truth-value assignment. Later, we shall see that this is exactly what extensionality with respect to a valuation requires. There is no "Carnap Problem" here. This is however, a case of a special implication relation on the sentences of classical sentential logic. The proof theory and this semantics are perfectly matched. Things don't always go this smoothly.

Consider the following structure: $\mathfrak{I}_{CSC}$, where S is the set of sentences of the Classical Sentential Calculus (CSC), and the implication relation $\Rightarrow$ is one given by say the standard deductive rules for (CSC). One would have thought that for such a familiar classical system, it would be obvious that the logical operators would all exhibit the same extensional pattern for every truth-value assignment. That is not so. We know that this structure has theses (those members of S which are implied by everything in the structure), that it has antitheses (those members of S which imply every member of the structure), and it is also incomplete in the sense that there are sentences in S such that neither they nor their negations are theses (this is sometimes call *syntactic incompleteness*, and sometimes *incompleteness with respect to negation*). Consider the following valuation: Let T = <K, L>, where L is the set of all theses, and K is the rest of S:

$\tau(A) = t$, if A is a thesis (A is in L), and
$\tau(A) = f$, if A is not a thesis (A is in K).

Let A_0 be a member of S such that neither it nor its negation is a thesis. Since the structure is classical, it follows that

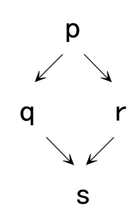
$\tau(A_0 \vee \neg A_0) = t$, and since neither A_0 nor $\neg A_0$ are theses,
$\tau(A_0) = \tau(\neg A_0) = f$.

Evidently under this assignment, a disjunction is assigned "t" although its disjuncts are both assigned "f". Furthermore there is a statement, A_0, such that it and its negation get assigned the same value. This is of course not your standard extensional distribution of truth-values.

We mentioned in passing that Carnap offered the example of a non-normal valuation that consisted in the assignment of "truth" to all sentences in a structure. In that case we get the bizarre distribution according to which every sentence and its negation are assigned the same value. Both these assignments satisfy the requirement that for any implication, if all the premises are "true" then so too is the conclusion. That looks like a "cheap shot". Nevertheless the response to such an example cannot be that we exclude such distributions, since that looks arbitrary. We have provided a less contentious example which supports Carnap's essential point.

In any case we cannot make use of Carnap's version since all the valuations provided by the Lindenbaum-Scott Completeness Theorem are based on (strong) bisections. If all sentences were assigned "t", then they would all belong to the set L, so that the set K would be empty. And that is impossible.

There are more exotic examples of which I shall consider just one. D. J. Shoesmith and T.J. Smiley give an example (*Multiple-Conclusion Logic*, p.3) in which there are four truth-values. It is possible to see their example as a case where the implication structure is given by a set S of four elements {p, q, r, s} where p implies q and implies r, q implies s, r implies s, and neither r nor s imply each other. In this case consider L to be the set {s}, and K to be the set {p,q,r}. Membership in L is "t", membership in K is "f". The structure looks like this:



In this four-membered Boolean algebra, s is the negation of p (and conversely), q is the negation of r (and conversely), the disjunction of q and r is s, so that the disjunction of two elements (q and r) that are "f", is "t", and q and its negation r are both “f”. Negation is classical. Church, in his review of Carnap’s *Formalization of Logic*, had already indicated that non-normal truth-value assignments could be based on a four-element Boolean algebra.

This example involves cases where there are various different kinds of falsity {p, q, r, }, and in such cases, one might be inclined to dismiss the example as a case of a generalization to multiple truth-values, and one might have expected with such a generalization, that distribution patterns of truth and falsity would lead to these peculiar results. That reaction however is not warranted. In order to see whether these non-normal interpretations are a serious problem requiring serious modification in the presentation of some of our standard logics, we have to have to have a better working definition of normal and non-normal assignments. That is the next task.

5. The Normal, and the Non-Normal

We can distinguish the normal truth-value valuations from those which are non-normal in a very simple systematic way, rather than appeal to a non-homogeneous collection of various clever, but strange constructions. Recall that a valuation on a structure $\Im = \langle S, \Rightarrow \rangle$ is a function which for any *strong* bisection <K, L> assigns "t" or "f" to a member of S according as it belongs to L or to K. And the bisection is strong just in case L is strongly closed under the implication relation ⇒. Recall as well, that by the Lindenbaum-Scott theorem, it is guaranteed that an implication relation holds between premises and a conclusion if and only if whenever any valuation is true for all of the premises, then it is also true for the conclusion.

Among all the valuations we shall single out those which are *normal*, for which we shall need the notion of the ***dual*** of an implication relation. It was R. Wojcicki (1973) who first defined the important notion of the dual of an implication relation ("Dual Counterparts of Consequence Relations", in *Bulletin of the Section of Logic*, v.2, n.1, 1973, pp. 54-57, and they have been studied at some length in Koslow (1992). Although Wojcicki seems not to have made any use of his discovery in his later writings, it is a seminal notion. Implication relations which are duals of other implication relations play a powerful role in defining a general notion of the duality of logical operators, and provide some insight into multiple conclusion logic as well. They also show that there are implication relations and consequence relations which are falsity-preserving rather than truth-preserving. [See also Koslow, *A Structuralist Theory of Logic* (1992).]

The dual of any implication relation is also an implication relation satisfying our (6) conditions, and it can be shown to satisfy two conditions: (1) A dually implies B (A ⇒^ B) if and only if B implies A (B ⇒ A), and although the dual is defined in all structures, in those structures where disjunctions exist, it reduces to this: (2) A finite number of premises dually implies B if and only if B implies their disjunction.

We now can introduce the notion of a *normal bisection*, which is a strong bisection with an additional condition: Let T = <K, L> be a bisection of an implication structure ℑ = <S, ⇒ >. Then T is a *normal bisection* on ℑ if and only if:

(1) L is strongly closed under the implication relation ⇒ of the structure, and

(2) K is strongly closed under ⇒^, the dual of the implication relation on the structure.

In the usual way, a *normal valuation* on the structure is one which for any normal bisection assigns "t" to the members of L, and "f" to those of K.

> A **non-normal valuation** on a structure with implication relation ⇒ is a valuation based on a strong bisection <L, K> for which (1) L is strongly closed under the implication relation, but (2) K is not strongly closed under the dual of that implication relation.

The restriction of valuations to normal ones introduces a nice symmetry in their construction: L is strongly closed under implication, and K is strongly closed under the dual implication. But there is more than just a symmetry that is reflected here. Normalcy requires that the concepts of true under a valuation and false under a valuation be duals of each other.

What we have in mind is the following: In the case of the logical operators on implication structures, the duals of operators can be obtained by taking their definitions which are framed in terms of implication, and simply replace the implication relation everywhere in that definition by it's dual. Thus for example, if in the characterization of conjunction we replace the implication relation by its dual, the result is the characterization of disjunction.

Similarly, we suggest, consider the characterizations we gave for truth (and falsity) in a normal valuation:

(1) For the normal bisection T = <K, L>, A is "t" if and only if (L is strongly closed under ⇒) and (A is in L).

And the dual would be given by

(2) For the normal bisection T = <K, L>, A is "f" if and only if (K is strongly closed under ⇒^) and (A is in K).

In other words, the assignment of falsity to the members of K is what the assignment of truth to the members of L becomes if we replace implication of the structure with its dual. Another way of seeing the connection with duality is to consider the simpler case: Let T = <K, L> be a strong bisection on a structure. The Ls are the truths, and the Ks are the false-hoods (for this bisection of course). For any member of L, anything which it implies (using ⇒) is true, and so in L. Now the Ks are false, and anything which implies them (using ⇒) is false, and so in K. However, any A which implies (using ⇒) some B in K is such that B dually implies A (B ⇒^ A). So K is closed under the dual implication. In effect, the same thing is going on, only in the one case it is by implication, and in the other it is by its dual.

Thus we see that the motivation to consider the normal valuations as the ones to use, is not to preserve some core logical truths like disjunctions being true if and only if at least one disjunct is true. There is the other possibility that the restriction to normal valua-tions respects a feature of truth and falsity: that they are dual concepts.

In certain recently studied logical systems, that duality has not been preserved. That doesn't mean that logic has been left in dire straits, and needs to be rescued. It only means that there are other paths that logicians can study and even advocate.

We can now see with this notion of normality in place, that ***if*** there are non-normal valuations on an implication structure, then there are going to be deviations from the usual distributional patterns for some of the logical operators.

Here is why we say "if": There are some implication structures on which all the valuations are normal, and there are some implication structures on which some valuations are normal and some are not. Here are some examples of these possibilities:

(1) If $\mathfrak{I}$ = <S, $\Rightarrow^{T}$ >, where the implication relation on S is a bisection implication relation on S, then it is easily shown that in that case where T = <K, L> is a bisection, then the valuation on it has to be normal. All the operators as a consequence have the familiar extensional distribution features. This was the example we already discussed on pp.123-4.

(2) Suppose that there is a non-normal valuation on an implication structure $\mathfrak{I}$ = <S, $\Rightarrow$ > in which disjunctions exist. Then there will be a disjunction such that the non-normal valuation will assign "f" to each of the disjuncts, but assign "t" to the disjunction. The proof is straightforward

> There is a strong bisection <K, L> in which L is closed under implication, but K is not closed under the dual implication. Then there will be some $A_1, A_2, ..., A_n$ and B in S, such $A_1, A_2, ..., A_n \Rightarrow^{\wedge}$ B, all the A_i are in K, but B is not. Then B $\Rightarrow (A_1 \vee A_2 \vee ... \vee A_n)$. B is in L, so $(A_1 \vee A_2 \vee ... \vee A_n)$ is also in L. Consequently we have a disjunction of members all assigned "f" by the non-normal valuation, but their disjunction is assigned "t". This shows that given our notion of a non-normal valuation, then in a very broad variety of cases, there will be Carnapian style examples of a non-standard distribution of truth-values.

(3) Here is a specific example of a non-normal valuation. Consider the classical implication structure (CSC) that we referred to earlier. Let <K, L> be a bisection where L is the set of all theses of (CSC), and K is the set of the remaining sentences of S (all the non-theses). (CSC) is incomplete (with respect to negation), so there is some sentence A_0 such that neither it nor its negation are theses of the structure. This is a non-normal bisection on the classical sentential calculus: L is certainly closed under the usual classical implication relation (say) $\Rightarrow$, but K is not closed under its dual. The reason is that $A_0, \neg A_0 \Rightarrow^{\wedge} (A_0 \vee \neg A_0)$ (because $(A_0 \vee \neg A_0) \Rightarrow (A_0 \vee \neg A_0)$). So we have A_0 is in K, and $\neg A_0$ is in K, but $(A_0 \vee \neg A_0)$ is in L. Thus with this non-normal valuation we have two statements each assigned

"f", whose disjunction is assigned "t", and a statement (i.e. A_0) such that it and its negation are both assigned "f".

Thus in classical implication structures, the distributional patterns of non-normal valuations for some of the logical operators deviate from the usual (extensional) patterns. This we have just seen is true for negation, and disjunction.

Before we turn to a way of getting some perspective on these observations, and try to understand the significance of the difference that non-normal valuations make, it is important to note that in the classical case, if we consider only the behavior of the normal valuations, then there is no departure from the familiar patterns. For this we need a brief discussion of the extensionality of the logical operators.

6. Extensionality and the Logical Operators

For any implication structure $\Im = \langle S, \Rightarrow \rangle$, we think of the logical operators as functions that map members of S, or pairs of members of S to S. The full story of how to define the logical operators using only the implication relation of the structure is a story told elsewhere in Koslow (1992). Suppose that one has an operator O(A) on the structure. Let T = <K, L> be any strong bisection on the structure. And let τ be the valuation based on that structure. We shall say that O(A) *is extensional with respect to the valuation* τ (for short, "**O[ext, τ]**"), if an only if ,

> For any A and A* in S, if τ(A) and τ(A*) are in the same set of the bisection (K, or L), then τ(O(A)) and τ(O(A*)) are in the same set of the bisection (K, or L).

That is, if A and A* have the same truth value, then O(A) and O(A*) also have the same truth-value. This definition covers the case of operators on single arguments. There is the obvious natural generalization for operators of two or more arguments.

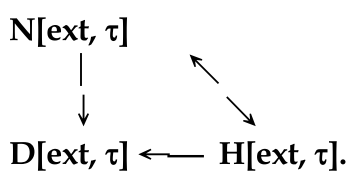
It can be shown that for any valuation τ, normal or not, and any A and B, (where "N", "D", "C", and "H", stand for the negation, disjunction, conjunction and conditional operators on a structure) that

(1) **N[ext, τ]** if and only if: ¬ A is in K if and only if A is in L.
(2) **D[ext, τ]** if and only if:(A ∨ B) is in L if and only if A is in L or B is in L.
(3) **C[ext, τ]** if and only if: (A ∧ B) is in L if and only if A is in L and B is in L.
(4) **H[ext, τ]** if and only if: A is in K or B is in L.

So (1) says that the negation operator ('¬") is extensional with respect to the valuation **τ,** if and only if [the negation of any A is assigned f if and only if A is assigned t]. Similar readings for (2) – (4).

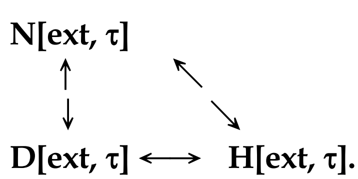
Therefore if a valuation departs from the customary distribution of truth-values for a logical operator, then that operator will fail to be extensional with respect to that valuation.

It is not difficult to show, for any implication structure, how the extensionality of the various logical operators with respect to any valuation (normal or not), are related. The result can be summed up this way:



So for example one can show that there is some valuation such that disjunction is extensional with respect to it, but negation is not.

The connection of the normality of a valuation and the extensionality of the various logical operators with respect to it is a matter of some delicacy. If we assume that the negation operator on an implication structure is classical, then it can be shown that all the logical operators on that structure are extensional with respect to any normal valuation—if any one of them is. That is:



Since it is straightforward to show that for any valuation τ, disjunction is extensional with respect to it, if and only if it is normal, it follows that in any classical implication structure, all the logical operators are extensional with respect to any normal valuation, since disjunction is extensional with respect to any normal valuation.

The matter is different if the implication structure is non-classical. Let $\Im_{ISC}$ be the structure that is associated with the Intuitionistic Sentential Calculus. It is easy to prove the following simple theorem:

> If $\Im$ is any implication structure such that (1) negation is non-classical, (2) it has the disjunctive property [(A ∨ B) is a thesis if and only if either A is a thesis or B is a thesis], and (3) $\Im$ is incomplete with respect to negation (some member of the structure is neither a thesis nor is its negation), then there exists a normal valuation on the structure such that negation is not extensional with respect to it.

This shows immediately that for the Intuitionistic implication structure, negation is not extensional with respect to some normal valuation, and by the first triangle diagram, the conditional is not extensional either. So the restriction to normal valuations, unlike the case of classical structures, doesn't help restore extensionality. The negation and conditional operators in the Intuitionistic structure are not -extensional even with respect to normal valuations.

> The proof is direct. Let T = <K, L>, where L is the set of Intuitionistic theses (all A such that it is provable in ISC that A). Clearly, L is closed under Intuitionistic implication $\Rightarrow^{ISC}$. To see that it is a normal bisection, consider A, B, C such that A, B $(\Rightarrow^{ISC})^\wedge$ C , where A and B are in K. We want to show that C is in K. Suppose it is in L, then since C $\Rightarrow^{ISC}$ (A v B) it follows that (A v B) is in L. Therefore by the disjunctive property, either A is in L or B is in L. But by hypothesis, neither of them is in L. Therefore C has to be in K. So T is a normal bisection. However, since ISC is incomplete with respect to negation, there is some Z such that neither it nor it's negation are theses. So neither it nor its negation are in L – both are in K. Therefore the valuation based on T assigns "f" to both Z and to its negation.

This is a nice way of incorporating what some philosophers think should be the Intuitionistic version of "truth" simpliciter. Here we take the appeal to the theses of Intuitionism to be one way of giving a truth-value interpretation—that is, a particular valuation. We have seen that taking the set of Intuitionistic theses as the L's of a strong bisection gives a nice example of a normal bisection. We do not confuse a particular truth-value assignment with "truth" (say some notion satisfying the Tarski T-Schema), no more than we would make the mistake of confusing a valuation in a two-valued sentential logic or a truth-value assignments for a possible world, with "truth".

So we see that a normal valuation on an Intuitionistic structure can give rise to deviations from the usual extensional distribution—in this case negation (and by the first triangle extensionality diagram the conditional will also fail to be extensional with respect to this normal valuation). The disjunction and conjunction operators, however, will be extensional with respect to this normal valuation.

It is also worthwhile mentioning the well known fact that just as there are non-normal valuations on classical structures, there are also non-normal valuations on non-classical structures. And just as in the classical case, they give rise to strange behavior for some logical operators. To see this, let $\mathfrak{I}_{ISC} = <S, \Rightarrow^{ISC}>$ be an Intuitionistic implication structure. Let T = <K, L> be a bisection where L is the set of *classical* theses (*sic*), and let K be the rest (S – L). L is closed under $\Rightarrow^{ISC}$, because if A is a classical thesis and A $\Rightarrow^{ISC}$ B, then B is also a classical thesis. So T is at least a strong bisection. The question is whether it is normal. Is K closed under the dual of the Intuitionistic implication relation $\Rightarrow^{ISC}$? The answer is negative.

> A proof: Since (CSC) is incomplete with respect to negation, there is some A_0 such that neither it nor its negation is a thesis of (CSC). Consequently neither of them is a thesis of ISC. Now since neither of them is a classical thesis, they are both in K. Now we have A_0 , $\neg A_0 \Rightarrow^{ISC}$ ^$(A_0 \vee \neg A_0)$, because that is equivalent to the condition that $(A_0 \vee \neg A_0) \Rightarrow^{ISC} (A_0 \vee \neg A_0)$. Therefore, if K is closed under the dual of $\Rightarrow^{ISC}$, then $(A_0 \vee \neg A_0)$ is in K. That is impossible since K contains only sentences which are not classical theses. Consequently, K is not closed under the dual, and this bisection is non-normal.

Therefore the valuation that is based on this strong bisection is non-normal. Since $\tau(A)$ is "t" if A is in L and "f" otherwise, we have the result that $\tau(A_0)$ and $\tau(\neg A_0)$ are both 'f". It is also clear from this proof, that disjunction and the conditional are also non-extensional with respect to this valuation. In any case, the restriction to only normal valuations may restore extensionality in the classical case, but it certainly doesn't achieve that in the Intuitionistic case, nor should it, since it is very clear that the negation and the conditional operators in the Intuitionistic case are easily seen to be non-extensional.

I agree with Church's reaction to the existence of non-normal valuations. He thought that Carnap had discovered that you could have valuations that assigned "t" to (say) the tautologies of the classical sentential calculus, but deviated elsewhere from the normal assignments on the logical operators. Van Fraassen's supervaluations are another very different way of showing that possibility.

In this note I have tried to indicate that the non-normal valuations have more interest than that. Their existence doesn't show that there is something wrong with the usual presentations of some simple sentential logical systems, nor does it show that the claim that implication is truth preserving is inadequate since it by itself doesn't guarantee the familiar extensional distribution of valuations on the logical operators. Nor does it guarantee that truth and falsity are duals. The motivation for removing non-normal valuations from the logical scene is cosmetic. There's no need to treat them like the lepers of logic.

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